Baby Jubjub Elliptic Curve

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Contents

1 Scope

This proposal aims to define a specific elliptic curve defined over the large prime subgroup of BN128 elliptic curve.

2 Motivation

The search for this elliptic curve defined is motivated by its usefulness in zk-SNARK proofs. Moreover the ability to find it in a deterministic way—so that it was clear no other considerations were taken for defining—is paramount as it significantly reduces the possibility of a backdoor being present, thus leading to better security.

3 Background

With this purpose, we used a deterministic algorithm for finding elliptic curves over a specified finite field [5] together with the restrictions of security parameters described in SafeCurves project [3].

4 Terminology And Description

4.1 Generation Of Baby Jubjub

In 2016, a group of researchers of IRPF designed a deterministic algorithm that, given a prime number p, it returns the elliptic curve defined over $\mathbb{F}_p$ with smallest coefficient A such that $A-2$ is a multiple of 4 and equation $y^2 = x^3 + Ax^2 + x$ describes a Montgomery curve. The assumption $A-2$ divisible by 4 comes from the fact that as this value is used in many operations, so trying to keep it smaller and divisible by four is a reasonable assumption [5].

SafeCurves is a project that checks some of the most common and known attacks on several elliptic curves. It also provides the algorithm it was used [3].

We considered the large prime number dividing the order of BN128 and run algorithm A.1 from [5]. The first elliptic curve it was returned satisfying SafeCurves criteria was the Montgomery curve with coefficient $A = 168698$. We named this curve Baby Jubjub elliptic curve.

4.2 Definition Of Baby Jubjub

From now on, let

$$p = 21888242871839275222246405745257275088548364400416034343698204186575808495617$$

and $\mathbb{F}_p$ the finite field with p elements.

4.2.1 Montgomery Form

We define E_M as the *Baby-Jubjub* Montgomery elliptic curve defined over $\mathbb{F}_p$ given by equation

$$E: v^2 = u^3 + 168698u^2 + u.$$

The order of E_M is $n = 8 \times r$, where

$$r = 2736030358979909402780800718157159386076813972158567259200215660948447373041$$

is a prime number. Denote by $\mathbb{G}$ the subgroup of points of order r, that is,

$$\mathbb{G} = \{ \, P \in E(\mathbb{F}_p) \mid rP = O \, \} \, .$$

4.2.2 Edwards Form

E_M is birationally equivalent to the Edwards elliptic curve

$$E: x^2 + y^2 = 1 + dx^2y^2$$

where $d = 9706598848417545097372247223557719406784115219466060233080913168975159366771.$

The birational equivalence [2, Thm. 3.2] from E to E_M is the map

$$(x, y) \to (u, v) = \left(\frac{1+y}{1-y}, \frac{1+y}{(1-y)x} \right)$$

with inverse from E_M to E

$$(u, v) \to (x, y) = \left(\frac{u}{v}, \frac{u-1}{u+1} \right).$$

4.3 Arithmetic In Baby Jubjub

In this section we define how to operate in the elliptic curve group: the addition of points and multiplication of a point by a scalar (an element of $\mathbb{F}_p$).

4.3.1 Addition Of Points

When adding points of elliptic curves in Montgomery form, one has to be careful if the points being added are equal (doubling) or not (adding) and if one of the points is the point at infinity [6]. Edwards curves have the advantage that there is no such case distinction and doubling can be performed with exactly the same formula as addition [2]. In comparison, operating in Montgomery curves is cheaper. In this section, we summarize how addition and doubling is performed in both forms. For the exact number of operations required in different forms of elliptic curves, see [2].

- <u>Edwards</u>: Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be points of the Baby-Jubjub twisted Edwards elliptic curve E. The sum $P_1 + P_2$ is a third point $P_3 = (x_3, y_3)$ with

$$\begin{aligned}
\lambda &= dx_1x_2y_1y_2,\\
x_3 &= (x_1y_2 + y_1x_2)/(1+\lambda),\\
y_3 &= (y_1y_2 - x_1x_2)/(1-\lambda).
\end{aligned}$$

Note that the neutral element is the point $O = (0,1)$ and the inverse of a point (x, y) is $(-x, y)$.

- Montgomery: Let $P_1 = (x_1, y_1) \neq O$ and $P_2 = (x_2, y_2) \neq O$ be two points of the Baby-JubJub elliptic curve E_M in Montgomery form.

  If $P_1 \neq P_2$, then the sum $P_1 + P_2$ is a third point $P_3 = (x_3, y_3)$ with coordinates

$$\begin{aligned}
\Lambda &= (y_2 - y_1)/(x_2 - x_1),\\
x_3 &= \Lambda^2 - A - x_1 - x_2,\\
y_3 &= \Lambda(x_1 - x_3) - y_1.
\end{aligned} \tag{1}$$

  If $P_1 = P_2$, then $2 \cdot P_1$ is a point $P_3 = (x_3, y_3)$ with coordinates

$$\begin{aligned}
\Lambda &= (3x_1^2 + 2Ax_1 + 1)/(2y_1),\\
x_3 &= \Lambda^2 - A - 2x_1,\\
y_3 &= \Lambda(x_1 - x_3) - y_1.
\end{aligned} \tag{2}$$

4.3.2 Multiplication Of A Point Of E By A Scalar

Let $P \neq O$ be a point of the Edwards curve E of order strictly greater than 8 (i.e. $P \in \mathbb{G}$) and let k a binary number representing an element of $\mathbb{F}_p$. We describe the circuit used to compute the point $k \cdot P$.

1. First, we divide k into chunks of 248 bits. If k is not a multiple of 248, we take j segments of 248 bits and leave a last chunk with the remaining bits. More precisely, write

$$k = k_0k_1 \dots k_j \quad \text{with} \quad \begin{cases} k_i = b_0^i b_1^i \dots b_{247}^i & \text{for } i = 0, \dots, j-1,\\ k_j = b_0^j b_1^j \dots b_s^j & \text{with } s \leq 247. \end{cases}$$

   Then,

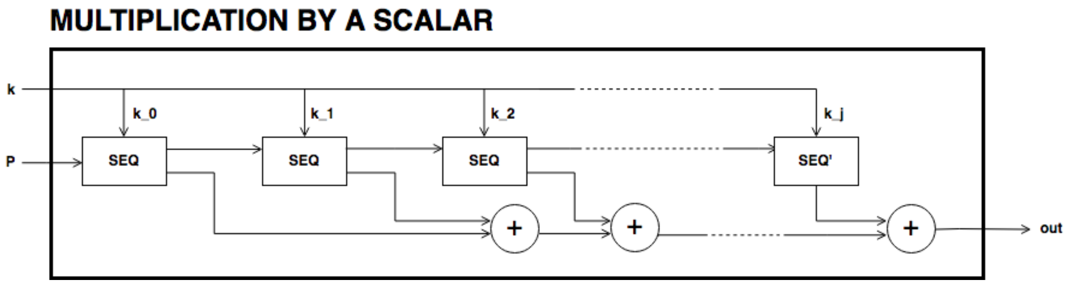
$$k \cdot P = k_0 \cdot P + k_1 \cdot 2^{248} P + \dots + k_j \cdot 2^{248j} P. \tag{3}$$

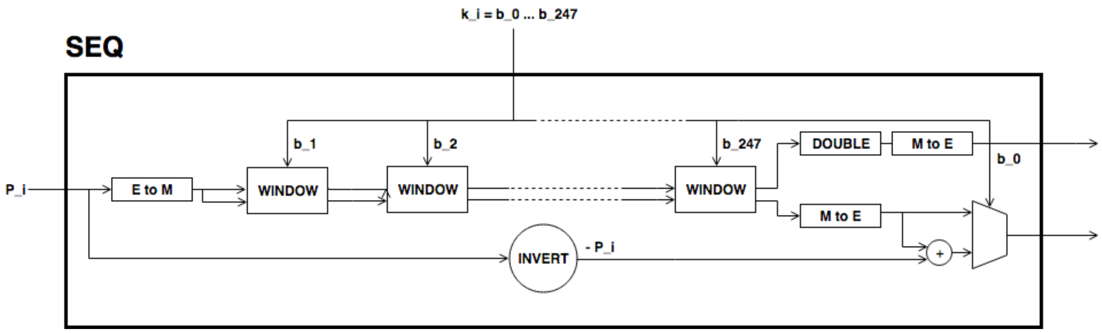
   This sum is done using the following circuit. The terms of the sum are calculated separately inside the SEQ boxes and then added together.

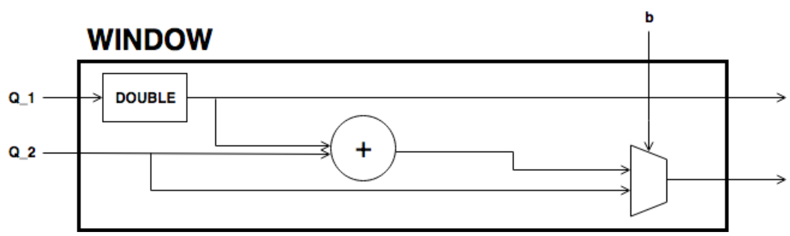
2. Each SEQ box takes a point of E of the from $P_i = 2^{248i}P$ for $i = 0, \dots, j-1$ and outputs two points

$$2^{248} \cdot P_i \quad \text{and} \quad \sum_{n=0}^{247} b_n \cdot 2^n \cdot P_i.$$

   The first point is the input of the next $(i+1)$-th SEQ box (note that $2^{248} \cdot P_i = P_{i+1}$) whereas the second output is the computation of the i-th term in expression (3). The precise circuit is depicted in next two figures SEQ and WINDOW.







The idea of the circuit is to first compute

$$Q = P_i + b_1 \cdot (2P_i) + b_2 \cdot (4P_i) + b_3 \cdot (8P_i) + \cdots + b_{247} \cdot (2^{247} P_i),$$

and output the point

$$Q - b_0 \cdot P_i.$$

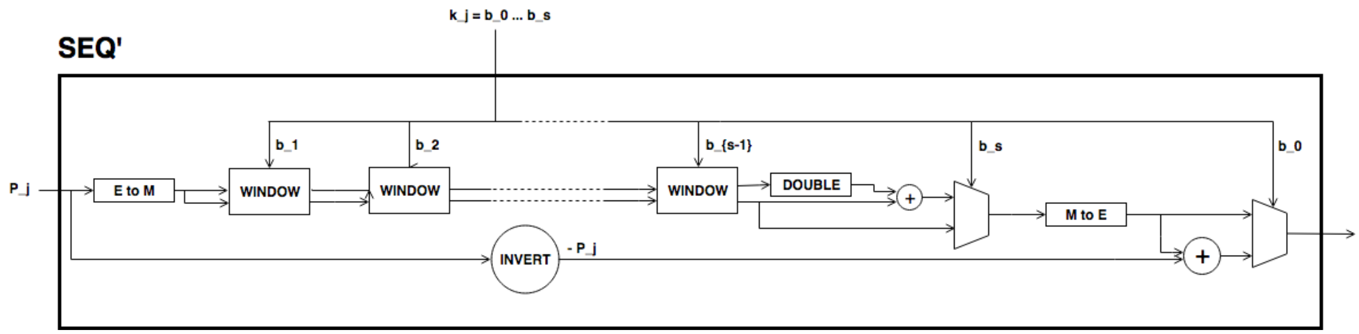
This permits the computation of Q using the Montgomery form of Baby-Jubjub and only use twisted Edwards for the second calculation. The reason to change forms is that, in the calculation of the output, we may get a sum with input the point at infinity if $b_0 = 0$.

Still, we have to ensure that none of the points being doubled or added when working in E_M is the point at infinity and that we never add the same two points.

- By assumption, $P \neq O$ and $\mathrm{ord}(P) > 8$. Hence, by Lagrange theorem [1, Corollary 4.12], P must have order r, $2r$, $4r$ or $8r$. For this reason, none of the points in E_M being doubled or added in the circuit is the point at infinity, because for any integer m, 2^m is never a multiple of r, even when 2^m is larger than r, as r is a prime number. Hence, $2^m \cdot P \neq O$ for any $m \in \mathbb{Z}$.
- Looking closely at the two inputs of the sum, it is easy to realize that they have different parity, one is an even multiple of P_i and the other an odd multiple of P_i, so they must be

different points. Hence, the sum in E_M is done correctly.

3. The last term of expression (3) is computed in a very similar manner. The difference is that the number of bits composing k_j may be shorter and that there is no need to compute P_{j+1}, as there is no other SEQ box after this one. So, there is only output, the point $k_j \cdot P_j = k_j \cdot 2^{248j}P$. This circuit is named SEQ'.



5 Challenges And Security

As required in the construction of Baby-Jubjub, the curve satisfies SafeCurves criteria. This can be checked following [4].

6 Implementation

Barry WhiteHat:

- `https://github.com/barryWhiteHat/baby_jubjub`
- `https://github.com/barryWhiteHat/baby_jubjub_ecc`

Jordi Baylina: `https://github.com/iden3/circomlib/blob/master/src/babyjub.js`

7 Intellectual Property

We will release the final version of this proposal under creative commons, to ensure it is freely available to everyone.

References

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[6] Okeya, K., Kurumatani, H., and Sakurai, K. Elliptic curves with the montgomery-form and their cryptographic applications. In *Proceedings of the Third International Workshop on Practice and Theory in Public Key Cryptography: Public Key Cryptography* (London, UK, UK, 2000), PKC '00, Springer-Verlag, pp. 238–257.